SOME RECENT DEVELOPMENTS IN RING THEORY

BY

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This article looks at three areas of ring theory where a decisive change in the subject has taken place. It is not intended to be comprehensive, but attempts to present the ideas in a reasonably self-contained form. The areas are discussed in three sections:

Section A. Rings with Krull dimension,

Section B. Rings with polynomial identity,

Section C. Universal enveloping algebras of Lie algebras.

Some introduction to terminology and theorems is to be found in [A3].

A. Rings with Krull dimension

1. Let R be a ring and M a right R-module. The *Krull dimension* $|M|$ of M is defined by induction. When $M = 0$, set $|M| = -1$ and for an ordinal α, knowing that $|M| \nless \alpha$, then set $|M| = \alpha$ provided that there is no infinite descending chain $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ of submodules such that $|M_{i-1}/M_i| \nless \alpha$.

There need be no such ordinal α, indeed its existence for M is an indicator of deviation from artinian modules, since M has minimum condition for submodules if and only if $|M| = 0$. The Krull dimension of a ring R is defined to be $|R_R|$. Strictly speaking, this is the *right* Krull dimension of R and, equally well, one can consider the left Krull dimension of R which may be different. For instance, there exist principal right ideal domains (right Krull dimension one) which are not left Ore, hence violate (A.05) and so do not have left Krull dimension. The idea originated in Gabriel-Rentschler [A2].

Some elementary properties are readily established (proofs in Gordon-Robson [A8]).

Let $0 \to K \to M \to N \to 0$ be exact; then:

(A.01) $|M| = \sup(|N|, |K|)$.

(A.02) $|R|$ = supremum of $|M|$ taken over finitely generated (cyclic is enough) modules.

(A.03) Krull dimension of a ring is a Morita invariant.

(A.04) A noetherian module has Krull dimension, Gabriel [A1].

(A.05) A module with Krull dimension has finite rank (no infinite direct sums of submodules).

Regarding Krull dimension of rings, it is readily proved that for a commutative noetherian ring the new Krull dimension coincides with the classical one given by chains of primes. This does not remain true without restriction. Take R to be the group ring of $\mathbb{Z}_{p\infty}$ over the field $\mathbb{Z}/(p)$. The augmentation ideal is the nil radical N and $N = N^2$ (check $N = N^p$). The ring is local and the classical Krull dimension is zero, but $|R|$ does not exist since $N = N^2$ does not hold with Krull dimension. See Goldie-Small [A5].

It is shown in [A2] that $|A_n| = n$, where A_n is the Weyl algebra. As A_n is a simple algebra the classical Krull dimension (defined by lengths of chains of primes) is zero. In particular, for A_1, we know that $|A_1| = 1$; this asserts that every proper factor module of R is artinian. In Hart [A10], [A11], the Krull dimension of simple Ore extension rings is considered:

$$R = K[x,\delta], \qquad xk - kx = \delta k, \qquad \delta: K \to K.$$

Here δ is a derivation and [A2] gives at once

$$|K| \leqq |R| \leqq 1 + |K|.$$

Now $|K| = |R|$ follows when the local ring K_p is regular at minimal primes. This holds when K is finitely generated over a field, due to the strong properties of δ which arise from the simplicity of R. See also Goodearl [A6] which deals with global dimension; the two concepts evidently have some unexplored connections. However, see the theorem of Roos mentioned in Björk [C1].

Other examples are given in [A8] together with the following somewhat surprising result:

(A.06). $R[x]$, with x a commuting indeterminate, has Krull dimension if and only if R has max-r.

To see this, take a chain of right ideals $I_0 \subset I_1 \subset \cdots$ of R, let

$$A = I_1 + I_2x + I_3x^3 + \cdots; \qquad B = I_0 + I_1x + I_2x^2 + ;$$

then

$$A/B \approx I_1/I_0 \oplus I_2/I_1 \oplus \cdots.$$

Then (A.05) shows that $\{I_n\}$ stops. The expected result (see [A8]) is then

(A.07) $$|R[x]| = 1 + |R|, \text{ when } R \text{ has max-r.}$$

An R-module M is a *critical module* if there exists an ordinal α such that $|M| = \alpha$ and $|M'| < \alpha$ for every proper factor module M'. Applied to the ring itself, it can be used to accept a right ideal I as critical, when $I = I_R$ is considered, or as critical when considering the factor module R/I. The former generalises the concept of *minimal* right ideal, the latter that of *maximal* right ideal. See [A4], [A10], [A8].

(A.08) A non-zero module M with Krull dimension has a critical submodule.

Take a non-zero submodule C with $|C|$ minimal, say $|C| = \beta$. If C is not critical, there is a submodule C_1 with $|C/C_1| = \beta$. Repeat for C_1 to obtain $|C_1/C_2| = \beta$ and so on. Then $C \supset C_1 \supset C_2 \supset \cdots$ is a chain which has to terminate because $|C| = \beta$. Let it end at C_k; then C_k is critical.

A module M can have critical submodules of different dimensions, say $|C_i| = \alpha_i$, where $\alpha_i \leqq |M|$. By (A.08) the minimum value for α_i is taken, but the existence of others is not fully understood. We also have:

(A.09) Let $|M| \geqq \alpha$ and S be the sum of all submodules $N \subset M$ with $|N| \leqq \alpha$. Then $|S| \leqq \alpha$.

This result, due to Gordon-Robson [A8], is very useful, its proof is complicated.

2. *Structure and applications.* Critical modules provide a valuable method for studying modules with Krull dimension (so including noetherian modules) which is similar in approach and feeling to the use of simple modules, composition series, and so on. See [A7] and [A5] for the following theorem.

THEOREM A.10. *A semi-prime ring with Krull dimension is a right order in a semi-simple Artin ring.*

PROOF. Let R have a critical right ideal C and take $0 \neq c_0 \in C$. The map $c \to c_0 c$, where $c \in C$, shows that

$$c_0 C \approx \frac{C}{r(c_0) \cap C}.$$

Evidently $r(c_0) \cap C \neq 0$ would imply that $c_0 C = 0$, because C is critical. Now let Z be the right singular ideal and choose $C \subseteq Z$. Since $r(c_0)$ is an essential submodule of R_R, then $r(c_0) \cap C \neq 0$ and $c_0 C = 0$. This holds for all $c_0 \in C$, hence $C^2 = 0$, $C = 0$ and $Z = 0$. Because (A.05) holds we can apply a result proved in [A3] to obtain the theorem.

A comparison with the usual Krull dimension $\mathrm{cl}|R|$ (lengths of chains of prime ideals, using ordinals if need be) is now possible.

$$|R| \geqq \mathrm{cl}|R|. \tag{A.11}$$

The result is due to Krause [A13] for fully bounded rings and extended to the general case in Goldie-Small [A5]. The dimensions usually differ for non-artinian rings, but they are equal for rings whose prime factor rings are bounded (see [A13]).

The possible coincidence of nil and nilpotent subrings has engaged ring theorists since the classical theorems of Hopkins for artinian rings and Levitski for noetherian rings. It became a natural problem for rings with Krull dimension, especially after Theorem (A.10) was proved. In Goldie-Small [A5] the problem was solved for commutative rings with finite Krull dimension; they also proved that the non-zero powers of the nil radical are different. Using this result and the concept of a localising subcategory, Robson-Gordon obtained the result for non-commutative rings with finite Krull dimension. Lenagan [14] proved this result independently by traditional methods. Finally these authors established the general theorem in [A9], [A15], [A8], again with different methods. Both proofs depend on (A.09).

THEOREM A.12. *Nil subrings of a ring with Krull dimension are nilpotent.*

The most remarkable application of Krull dimension is the proof by Jategaonkar in [A12] that the intersection of the powers of the Jacobson radical of a fully bounded noetherian ring (FBN-ring) is zero. This classical problem was propounded by Jacobson in 1944; Herstein (see Section B) gave a counter-example for the case of rings with max-r only. A ring is fully bounded when all prime factor rings are bounded (each essential right ideal contains a two-sided ideal) and noetherian means max-r and max-l.

In particular the theorem holds in a noetherian ring with polynomial identity; Herstein's example shows that max-r is not enough.

THEOREM A.13. *Let R be an* FBN-*ring and J be its Jacobson radical. Then*

$$J^{\infty} = \left(\bigcap_{n=1}^{\infty} J^{n}\right) = 0.$$

The proof in [A12] is rather involved and little can be said here about details. A theory is developed for modules with Krull dimension which generalises the notions of composition series and Jordan-Hölder theorems in which simple factors are replaced by α-critical modules (varying ordinals α). A module is α-smooth if it has a finite series whose factors are α-critical for fixed α. In particular the following holds.

(A.14) Let R be an FBN-ring. A finitely generated R-module is α-smooth if and only if this holds for all finitely generated submodules. An essential extension of an α-smooth module is α-smooth.

The important case for the theorem is the following.

(A.15) Let M be a module which has a unique minimal submodule; then M has a composition series.

Now take any right ideal V_a of R which does not contain an assigned element $a \in R$ and is maximal with respect to this property. Apply (A.15) to the factor module R/V_a. It follows that R/V_a has a composition series and then $J^n \leqq V_a$ for some $n > 0$. Now $JJ^\infty = \bigcap V_a$ taken over all $a \notin JJ^\infty$. It follows that $J^\infty = JJ^\infty$ and $J^\infty = 0$ by Nakayama's lemma.

B. Rings with polynomial identity

In recent years there has been remarkable progress in this field, some earlier problems have been solved (not always as expected). and there has been a striking application to classical algebra in the discovery by Amitsur of non crossed-product division algebras. We can only give a sketch of these results in the space available. It is salutary to try a comparison with the state of the theory reported in [A3]. See also Herstein [B8] and Jacobson [B10] for more basic information.

One early question asked by Kaplansky when he initiated PI theory was to what extent is it a theory of subrings of matrix rings C_n, over commutative rings C. This is true for rings without nil ideals (see [B8]) but the general case was settled much later in a counter example due to Small [B16]. The example rests on the following argument. Let R be a finitely generated PI algebra over a field F and let $R \subset C_k$, the ring of $k \times k$ matrices over a commutative ring C. Then

$$R \approx F[x_1, \cdots, x_n]$$

where the x_i are matrices from C_k. Taking $\bar{C}$ to be the algebra generated by the entries of the x_i, this is a noetherian ring; then clearly $R \subset \bar{C}_k$. It follows that R has the ascending chain condition (ACC, and by symmetry DCC also) for right annihilators. See [A3] for some definitions. Small finds a nice example of a PI algebra without this property. Let A be the ring of matrices of the form

$$\begin{pmatrix} F[t,t^{-1}] & F[t,t^{-1}] \\ 0 & F[t] \end{pmatrix}$$

where F is a field and t an indeterminate over F. Then A is a finitely generated algebra over F, it has max-l (but not max-r) and lies in the matrix algebra $(F(t))_2)$. Let I be the right ideal of A generated by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Now go to $B \subset A_2$ consisting of matrices $\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}$, with a scalar matrix a_1 and a_2, $a_3 \in A$. So $B = \begin{pmatrix} F & A \\ 0 & A \end{pmatrix}$; now take the ideal $\hat{I} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ and set $R = B/\hat{I}$. Then R is a finitely generated algebra over F, it satisfies all identities of F_4, and does not have DCC on right annihilators. By going to $F_4 \oplus R$, the example can be stretched to have a nilpotent Jacobson radical yet retains its inability to be a subring of C_k. This example disposes of several problems, Kaplansky [B11], Amitsur and Procesi [B1].

Bergmann has since obtained a *finite* ring which is a counter-example, namely $R = \mathrm{End}\ (\mathbb{Z}_p \oplus \mathbb{Z}_{p^2})$ where p is a prime. See [B5]. However, the first example uses a technique which is at the heart of present PI theory, namely the correlation of a ring of matrices with the noetherian ring of its entries.

Bergmann has also given in [B5] an example of a semi PI ring which does not have a classical quotient ring. This problem had arisen naturally, when Posner's theorem appeared. The example is a subdirect sum of copies of the matrix ring F_k over a field F, defined by geometric considerations. The question of the structure of semi-prime PI rings is still obscure.

Let Ω be a commutative ring. A finite set $X_1, \cdots, X_m$ of $k \times k$ matrices is *generic over* Ω if its entries are mk^2 commuting indeterminates over Ω, and

$$\Omega[X_1, \cdots, X_m]$$

is the *algebra of* $k \times k$ *generic matrices over* Ω. The generic algebra is a subalgebra of $(\Omega[\xi_i])_k$ where the ξ_i run over the mk^2 entries concerned. This algebra is the foundation of most recent work in PI theory.

It was shown by Amitsur [B2] that a central division algebra exists which is not a crossed product; this settles a long outstanding problem. Let F be a field and $F[n; X_1, \cdots, X_m]$ be the m-fold algebra of $n \times n$ generic matrices over F. Amitsur showed that this algebra is an integral domain which has a classical ring of quotients

$$F(n; X_1, \cdots, X_m).$$

The latter is a division algebra of dimension n^2 over its centre, since it satisfies exactly the polynomial identities of $n \times n$ matrices (need $m > 1$). See Amitsur [B2] for the following.

(B.01) Let Q be the rational field; then $Q(n; X_1, \cdots, X_m)$ is not a crossed product, when n is either divisible by 8 or by the square of an odd prime.

Using (B.01), but transferring to characteristic p, Schacher and Small [B15] have another range of examples.

(B.02) Let F be a field of characteristic p which is not algebraic over $\mathbb{Z}_p$. Let $n \in Z^+$ and $(n, p) = 1$. Then $F(n; X_1, \cdots, X_m)$ $(m > 1)$ is not a crossed product if n is divisible by the cube of a prime.

Details of the proofs of these results are rather complicated but in common with many parts of PI theory they employ a simple valuable technique of localisation due to Small [B16].

(B.03) Let R be a prime ring and P a prime ideal of R such that the minimal degree of identities for R/P equals that of R. Let $\mathscr{C}(P)$ be the set of elements of R which are regular modulo P. Then $\mathscr{C}(P)$ is a localising set of regular elements of R and the local ring $R_{\mathscr{C}(P)}$ has a unique maximal ideal $PR_{\mathscr{C}(P)}$.

The condition on P really asks that if $S_{2n}(x) = 0$ in R/P then the same identity holds in R. Also, $\mathscr{C}(P)$ is localising means that for any $a \in R$, $c \in \mathscr{C}(P)$ there exist $a_1 \in R$, $c_1 \in \mathscr{C}(P)$ with $ac_1 = ca_1$. The local ring is formed in the usual way (see [A3]) and satisfies $S_{2n}(x) = 0$.

Since Formanek showed the existence of central polynomials (see [B7]) this technique can be replaced by central localisation, as has been done by Rowen in [B14].

The polynomials discovered by Formanek are universal in nature, their coefficients are integers (including ± 1), and they are central for any matrix ring C_n where C is a commutative ring. A *universal central polynomial* is an element $p(X_1, \cdots, X_k)$ of the free algebra $\mathbb{Z}[X_1, \cdots, X_k]$ such that for any C_n (n assigned), $p(a_1, \cdots, a_k) \in C$ (actually is a scalar matrix) whenever the $a_i \in C_n$, and is non-zero for some choice of the a_i. Clearly some coefficient of the polynomial has to be ± 1. The Formanek construction is explicit:

(B.04) Let $x_1, \cdots, x_{n+1}$ be commuting variables and X, $Y_1, \cdots, Y_n$ non-commuting variables. Use the map

$$x_1^{a_1} \cdots x_{n+1}^{a_{n+1}} \to x^{a_1} y_1 x^{a_2} y_2 \cdots y_n x^{a_{n+1}}$$

to transform

$$g(x_1,\cdots,x_{n+1}) = \prod_{2\leqq i\leqq n}(x_1 - x_i)(x_{n+1} - x_i)\prod_{2\leqq k<k\leqq n}(x_j - x_k)^2$$

into $G(X, Y_1,\cdots, Y_n)$, coefficients of monomials being unaltered.

Then $G(X, Y_1,\cdots, Y_n) + G(X, Y_2,\cdots, Y_n, Y_1) + G(X, Y_n, Y_1,\cdots, Y_{n-1})$ is a universal central polynomial for $n \times n$ matrices.

For a variety of reasons, it is natural to suppose that the prime factor rings should be important for the structure of a PI ring. However, the minimal degree of identities may go down under homomorphic images, an aspect of the theory which is little understood. More is known when it does not, this is due to a remarkable theorem of M. Artin [B4].

(B.05) Let R be a K-algebra, where K is a field; then R is an Azumaya algebra of rank n^2 over its centre if and only if:

(i) R satisfies all the identities of K_n;

(ii) no factor ring of R satisfies an identity of K_{n-1}.

For the properties of Azumaya algebras see [B6] and [B12]; they are also termed central separable algebras. The theorem can be thought of as a generalisation of Kaplansky's theorem that a primitive PI ring is a central simple algebra, since central separable and central simple coincide for algebras over fields.

This result has been extended to rings by Procesi in [B13]. A ring R is said to be a $\mathbb{Z}_n$-ring if it satisfies (i) and (ii) with $\mathbb{Z}$ replacing K. Then:

(B.06) A ring is a $\mathbb{Z}_n$-ring if and only if it is an Azumaya algebra of rank n^2 over its centre.

The difficult part of this theorem rests in proving that a $\mathbb{Z}_n$-ring R is Azumaya. The relevant properties at which to aim are that R is f.g. projective over its centre Z and that R/MR is central simple of dimension n^2 over its centre Z/M for any maximal ideal M. A classical argument is needed and some form of localisation. Certainly that of (B.03) is available at every prime ideal of R and was used by Procesi. However, later proofs due to Amitsur [B] and Rowen [B14] rely more on the Formanek polynomials. In particular Amitsur uses the Formanek construction to obtain new classes of central polynomials for central simple algebras.

The theorem can be used to study conditions under which the intersection of the powers of the Jacobson radical J of a PI ring is zero. The theorem of Jategaonkar [A12] shows that $J^\infty = 0$ for a PI ring with max-r and max-l. It is evident that the required condition of boundedness is satisfied (for example, see the proof of

(B.03) in [B16]), Moreover, Herstein gave an example of a PI ring with max-r but not max-l such that $J^\infty \neq 0$.

(B.07) Let $S = \begin{pmatrix} \mathbb{Z}_p & Q \\ 0 & Q \end{pmatrix}$, where Q is the rational field, and $\mathbb{Z}_p$ denotes localisation of $\mathbb{Z}$ at the prime p. Then $\bigcap_{n=1}^\infty J(S)^n \neq 0$. S has PI and max-r.

However, Herstein and Small [B9] have shown that progress can be made in these difficult circumstances, using Artin's theorem. Define a sequence of infinite intersections

$$J_1 = \bigcap_{n=1}^\infty J^n, \quad J_2 = \bigcap_{n=1}^\infty J_1^n, \quad \text{etc.}$$

(B.08). Let R be a PI ring with max-r; then $J_k = 0$ for some integer k.

It suffices to prove the theorem when R is a prime ring for the nil radical N is an intersection of minimal primes

$$N = P_1 \cap \cdots \cap P_m,$$

and if $J_{k_i} \subseteq P_i$ then $J_k \subseteq N$ where $k = \max(k_1, \cdots, k_m)$.

Thus $J_{k+1} = 0$; indeed, $J_k^\rho = 0$ since $N^\rho = 0$ by Levitski's theorem. In the prime case, the simplest position for proving the theorem occurs when the Formanek centre (the set of central values taken by all Formanek polynomials; it is a subring of the centre of R) has an element a not in J. Then $R[a^{-1}]$ is an Azumaya algebra and one can use the 1–1 correspondence between its ideals and those of its centre. This would in fact give $J_1 = 0$, but the method only gives this locally.

However, it is still uncertain whether $J_1 = 0$ for a prime PI ring with max-r, which would imply $J_1^\rho = 0$ when primeness is dropped. The problem has been open for some time.

Another application of Artin's theorem is given in Small [B18], again making use of the 1–1 correspondence between the ideals of an Azumaya algebra and those of its centre.

(B.09) If R is a PI ring with the maximum condition on two-sided ideals then R has the descending chain condition on prime ideals.

C. Universal enveloping algebras of Lie algebras

Progress has been made with the elusive structure of the Weyl algebras A_n. We remind the reader that A_1, or $A_1(k)$, is defined to be the free k-algebra $k[x, y]$ subject to $xy - yx = 1$, where k is the base field and $A_n = A_1 \otimes \cdots \otimes A_1$, taken

n times over k. These algebras are simple noetherian domains and have an intimate relationship with the universal enveloping algebras of Lie algebras. (See [A3] for earlier work.) The automorphism group of A_1 has been analysed in detail by Dixmier, using a detailed classification of the forms of the elements of A_1. (See [C7], [C8].) Note that k always has zero characteristic.

Modules of finite length are studied by McConnell-Robson [C17].

(C.01) $\mathrm{Hom}_A(C, D)$ and $\mathrm{Ext}^1_A(C, D)$ are finite dimensional vector spaces over k, where $A = A_1 = A_1(k)$ is the Weyl algebra and C, D are non-free cyclic A-modules.

This result is obtained by a form of localisation, which relates A_1 to a pair of quotient rings each being a (non-commutative) principal ideal domain. These rings all lie in the classical quotient ring of A_1. The same method is employed in Roos [C24] to obtain:

(C.02) $$\mathrm{gld}\, A_n(k) = n.$$

This settles a well-known problem initiated by Rinehart [C23]. It has been generalised to algebras of differential operators by Björk [C1].

The ideal structure of enveloping algebras has been studied for several years (see [A3] for the earlier work). Taking the Lie algebra $\mathfrak{g}$ to be nilpotent over an algebraically closed field k, Dixmier [C5] showed that there is a bijection between the sets of primitive ideals and orbits of G on $\mathfrak{g}^*$. Here $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$ and G is the smallest algebraic group of automorphisms of $\mathfrak{g}$ whose Lie algebra contains $\mathrm{ad}\mathfrak{g}$. For nilpotent Lie *groups* the corresponding connection between simple representations and orbits was established much earlier by Dixmier and Kirillov.

The details of the bijection are as follows. Let $\lambda \in \mathfrak{g}^* = \hom(\mathfrak{g}, k)$ and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is *subordinate* to λ if λ restricted to $\mathfrak{h}$ is a one-dimensional representation of $\mathfrak{h}$, so that $[\mathfrak{h}, \mathfrak{h}] \subset \ker \lambda$. Taking $\mathfrak{h}$ to be subordinate to λ and of maximal possible dimension, let $I(\mathfrak{h}, \lambda)$ be the left ideal of the universal enveloping algebra $U = U(\mathfrak{g})$ generated by the set $\{h - \lambda(h) \mid h \in \mathfrak{h}\}$ and let $P(\mathfrak{h}, \lambda)$ be the *bound* of $I(\mathfrak{h}, \lambda)$, namely the largest ideal in $I(\mathfrak{h}, \lambda)$. Then $P(\mathfrak{h}, \lambda)$ is a primitive ideal of U and $P(\mathfrak{h}, \lambda) = P(\mathfrak{h}', \mu)$ if and only if λ, μ belong to the same orbit of $\mathfrak{g}^*$ under G.

When $\mathfrak{g}$ is solvable, Dixmier [C6] proved in essence that this map from the set $\mathfrak{g}^*/G$ of orbits of G on $\mathfrak{g}^*$ to the set of primitive ideals is surjective. Rentschler [C22] has shown recently that it is bijective. Taking the topology of $\mathfrak{g}^*/G$ to be the quotient of the Zariski topology on $\mathfrak{g}^*$ and that for the primitive ideals to be the usual Stone topology on $\mathfrak{g}^*$ (see Jacobson [B10]) it is natural to expect the correspondence to be a homeomorphism. For nilpotent Lie *groups* the corresponding possibility was raised by Kirillov in 1962 and settled affirmatively by Ian Brown [C3]. Recently N. Conze [C4] has adapted the arguments used by Brown to pass from the group to the algebra case and obtained:

(C.03) The correspondence $g^*/G \to \text{prim } U$ is a homeomorphism when $\mathfrak{g}$ is a nilpotent Lie algebra.

When $\mathfrak{g}$ is solvable, it is only known that this correspondence is a piecewise homeomorphism. See [C2].

Now let $\mathfrak{g}$ be *completely solvable* (i.e., $\mathfrak{g}$ has a chain of ideals $0 = \mathfrak{g}_0 < \mathfrak{g}_1 < \cdots < \mathfrak{g}_m = \mathfrak{g}$ with $\dim \mathfrak{g}_i = i$; by Lie's theorem these include solvable algebras over algebraically closed fields). Suppose that P is a prime ideal of $U(\mathfrak{g})$ and let E be the set of elements of U/P which generate one-dimensional $\mathfrak{g}$-submodules of U/P, under the action $g[u+P] = [[g,u]+P]$. Then U/P has a localisation $(U/P)_E$. In fact, when $\bar{g} = [g+P]$ and $e \in E$, then

$$\bar{g}e - e\bar{g} = \lambda_g e \text{ for some } \lambda_g \in k.$$

The Ore condition takes the form $\bar{g}e = e(\bar{g}+\lambda_g)$ for these generators and for general elements gives $\bar{U}e = e\bar{U}$, where $\bar{U} = U/P$. This shows that the elements of $(U/P)_E$ are quotients ae^{-1}, so it is a subring of the quotient division ring of U/P. Moreover $(U/P)_E$ is a simple ring.

When P is primitive and M is a simple $\mathfrak{g}$ (equally $U(\mathfrak{g})$)-module with $\operatorname{ann} M = P$, then M is a $(U/P)_E$-module. So the simple modules for $U(\mathfrak{g})$ with kernel P coincide with the simple $(U/P)_E$-modules. Results obtained are:

(C.04) (Dixmier [C5] and Nouazé-Gabriel [C20].) Let $\mathfrak{g}$ be nilpotent; then E is the centre of U/P and $(U/P)_E \approx A_n(K)$, where K is the centre of $Q(U/P)$.

Thus central localisation acts as an intermediate quotient ring; a different story from the PI case, where Formanek's polynomials and Posner's theorem would give the full quotient ring. The latter situation occurs for all U/P, when $\mathfrak{g}$ is nilpotent and k has *prime characteristic.*

When $\mathfrak{g}$ is solvable McConnell [C18] has shown that $(U/P)_E$ is an algebra of differential operators on a commutative algebra with multiplication changed by a 2-cocycle. Later [C19] he obtained a more transparent representation of these algebras.

Consider $A_l \oplus U(w)$, where A_l is the Weyl algebra

$$A_l = k[x_1, y_1; \cdots; x_l, y_l]; \qquad x_i y_i - y_i x_i = 1,$$

and $U(w)$ is a commutative polynomial algebra (thus w is an abelian Lie algebra). Let

$$V = w + \sum_{i=1}^{l} (kx_i + ky_i).$$

Choose $\lambda \in V^* = \hom(V, k)$. This defines a unique automorphism θ_λ of the algebra $A_l \otimes U(w)$ given by

$$x_i \to x_i + \lambda(x_i) \colon y_i \to y_i + \lambda(y_i) \colon w \to w + \lambda(w), \qquad w \in w.$$

Let G be a finitely generated subgroup of V^*; then G is free abelian and the map $\lambda \to \theta_\lambda$ defines a group monomorphism

$$G \to \mathrm{Aut}(A_l \otimes U(w)).$$

Let $\mathcal{G}$ = image G under this map. The twisted group ring

$$(A_l \otimes U(w)) \# k\mathcal{G}$$

is simple if and only if $w \cap (\bigcap_{\lambda \in G} \ker \lambda) = 0$.

Note that $R \# k\mathcal{G}$ for a k-algebra R is defined linearly as $R \otimes_k k\mathcal{G}$ but with multiplication generated by

$$(r \otimes g)(r' \otimes g') = r\sigma_g(r') \otimes gg',$$

where $g \to \sigma_g$ is an assigned homomorphism $\mathcal{G} \to \mathrm{aut}\, R$.

(C.05) (McConnell [C19].) Let P be a prime ideal of $U(g)$, where g is completely solvable. Then if K is the centre of the simple algebra $(U/P)_E$, one has

$$(U/P)_E \approx (A_l \otimes U(w)) \# K\mathcal{G} \tag{C.06}$$

for suitable choice of parameters. Conversely, any simple algebra of the form (C.06) is isomorphic to $(U/P)_E$ for some abelian-by-abelian Lie algebra g. Thus the simple representations of solvable Lie algebras can be studied through the abelian-by-abelian case.

As an illustrative example, take $U = k[u, v]$ with $uv - vu = u$; thus g is 2-dimensional solvable. Then $E = \{u^n : n = 1, 2, \cdots\}$ and $U_E = k[u, u^{-1}, v]$. Then $U_E \approx U(kv) \# k\mathcal{G}$, where $\mathcal{G}$ is infinite cyclic. Now U_E is also isomorphic to

$$k[x, x^{-1}, y], \; xy - yx = 1, \quad \text{under} \quad u \to x, \, u^{-1}v \to y,$$

so U_E is a partial quotient ring A_1' of A_1.

Let A'_m denote the mth tensor power of A'_1 over k.

In [C14], Gelfand and Kirillov conjectured that if k is algebraically closed and g is an algebraic Lie algebra then K = centre $Q(U(g))$ is a pure transcendental extension of k and there exists $n \geqq 0$ such that

$$Q(U(g)) \approx Q(A_n(K)).$$

For solvable Lie algebras this is answered by McConnell [C18] and Borho [C2].

(C.07) Let g be algebraic and completely solvable. Then

$$(U/P)_E \approx A_l(K) \otimes_K A'_m(K),$$

where K = centre $Q(U/P)$ and m, $n \geqq 0$.

Thus $Q(U/P) \approx Q(A_{l+m}(K))$ and in case $P=0$, Bernat (see [C2]) has shown that K is a pure transcendental extension of k. That $Q(U) \approx Q(A_n(K))$ has also been proved by Joseph [C16].

More generally we have:

(C.08) (McConnell [C19].) Let g be completely solvable. If

$$(U/P)_E \approx (A_l(\otimes U(w)) \# K\mathscr{G},$$

as in (C.05), then $(U/P)_E$ is a subalgebra of $A_l \otimes A'_m$, where m = rank $\mathscr{G}$.

Gelfand-Kirillov in [C15] have partially solved their conjecture for the semi-simple algebras by proving that a suitable enlargement of the centre of U gives an extension ring of $U(g)$ which has its quotient division ring of the form $Q(A_n(K))$.

Returning to the study of primitive ideals P of $U(g)$, let k be algebraically closed (remember also of zero characteristic) then $P \cap Z$ is a maximal ideal of the centre Z of $U(g)$. See [C5], [C21].

(C.09) (Dixmier [C9].) Let g be semi-simple and M be a maximal ideal of Z; then the set of primitive ideals P of $U(g)$ such that $P \cap Z = M$ is a finite set with unique maximal and minimal elements.

Refer also to [A3, Sect. 4], for earlier results known for nilpotent and solvable algebras. They include the knowledge that, for g nilpotent, $U(g)$ has but a finite number of simple factor rings (to isomorphism), namely A_n with $n < \frac{1}{2}\dim g$. Dixmier in [C10] shows that the semi-simple case is far more complex. Even when $g = Sl(2,\mathbb{C})$, the complex simple Lie algebra of dimension three, $U(g)$ has an infinite number of non-isomorphic simple factor algebras, even restricting to domains, the so-called B_λ introduced in [C10]. These are closely related to A_1.

Finally, a striking result, Duflo [C12].

(C.10) $U(\mathfrak{g})$ for arbitrary $\mathfrak{g}$ is a Jacobson ring.

This was proved in the solvable case by Dixmier in [C6] and he then showed that a prime ideal is primitive if and only if the centre of $Q(U/P)$ is an algebraic extension field of k. A restriction to k, that it be algebraically closed and non-denumerable can be removed by the lemma of Quillen [C21].

Finally, the earlier-mentioned correspondence between orbits in $\mathfrak{g}^*$ and primitive ideals of $U(\mathfrak{g})$ has been shown by Duflo [C13] to yield a dense set of primitives even for arbitrary finite dimensional $\mathfrak{g}$.

I am indebted to John McConnell for providing the information given in Section C. The book J. Dixmier [C11] provides a full account of these results.

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